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Translated by N. H. C.

UDC 531.36

ON STABILITY OF MOTION WITH RESPECT TO A PART OF VARIABLES IN THE CRITICAL CASE OF A SINGLE ZERO ROOT

PMM Vol. 39, № 3, 1975, pp. 422-426
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(Received July 8, 1974)

The stability of motion with respect to a part of variables is investigated for the critical case of a single zero root. The criterions of stability and instability are obtained.

1. Consider the system of differential equations of perturbed motion

$$dx_i/dt = X_i(x_1 \ldots, x_n), \qquad i = 1, \ldots, n \tag{1.1}$$

We shall investigate the problem of stability of unperturbed motion $x_i = 0$ $(i = 1, \ldots, n)$ with respect to $x_1, \ldots, x_m$ $(m > 0, \quad n = m + p, \quad p > 0)$. We denote these variables by $y_i = x_i$ $(i = 1, \ldots, m)$, and the remaining ones by $z_j = x_{m+j}$ $(j = 1, \ldots, p)$ [1, 2]. Let the functions X_i represent power series expanded in the powers of y_i $(i = 1, \ldots, m)$ and z_j $(j = 1, \ldots, p)$ and convergent in the region

$$|y_i| \leqslant h, \quad i = 1, \ldots, m, \quad |z_j| \leqslant H < \infty, \quad j = 1, \ldots, p \tag{1.2}$$

where h and H are certain constants.

Now the equations of perturbed motion (1.1) assume the form

$$\frac{dy_i}{dt} = \sum_{j=1}^{m} a_{ij} y_j + Y_i(y_1, \ldots, y_m, z_1, \ldots, z_p), \quad i = 1, \ldots, m \tag{1.3}$$

$$\frac{dz_k}{dt} = \sum_{j=1}^{m} b_{kj} y_j + \sum_{j=1}^{p} c_{kj} z_j + Z_k(y_1, \ldots, y_m, z_1, \ldots, z_p), \quad k = 1, \ldots, p$$

where a_{ij}, b_{kj} and c_{kj} are constants, Y_i and Z_k are functions of the variables $y_1, \ldots, y_m, z_1, \ldots, z_p$ which are expanded in the region (1.2) into power series in these variables with the first terms of at least second order. The variables z_j $(j = 1, \ldots, p)$ are always bounded (Condition A), since they belong to the region (1.2). This condition represents the starting assumption in the investigation of (1.3).

Let

$$Y_i(0, \ldots, 0, z_1, \ldots, z_p) = 0, \quad |Y_i| \leqslant \sum_{j=1}^{m} h_{ij} |y_j|, \quad i = 1, \ldots, m \tag{1.4}$$

where h_{ij} are sufficiently small positive constants.

Let us explain the conditions of stability and instability in the first approximation with respect to the variables $y_1, \ldots, y_m$ for the case when the equations of perturbed motion have the form (1.3). According to [3], the following theorem holds.

Theorem 1. If all roots of the equation

$$| a_{ij} - \delta_{ij} \lambda | = 0 \tag{1.5}$$

have negative real parts, then the unperturbed motion of the system (1.3) is, with the Condition A holding, asymptotically stable with respect to the variables $y_1, \ldots, y_m$ no matter what are the higher order terms appearing in the first group of Eq. (1.3) and satisfying the conditions (1.4).

The following theorems are also valid.

Theorem 2. If the roots of the equation (1.5) contain at least one root with the positive real part, then the unperturbed motion of the system (1.3), with the Condition A holding, is unstable with respect to the variables $y_1, \ldots, y_m$ no matter what are the higher order terms appearing in the first group of equations (1.3) and satisfying the condition (1.4).

Proof. Consider the quadratic form $v(y_1, \ldots, y_m)$ defined by the equation

$$\sum_{i=1}^{m} \frac{\partial v}{\partial y_i} \Big(\sum_{j=1}^{m} a_{ij} y_j \Big) = \alpha v + \sum_{i=1}^{m} y_i^2$$

where α is a positive number. Such a form v naturally exists, and can assume positive values [4]. Let us write the time derivative of this form by virtue of (1.3)

$$\frac{dv}{dt} = \alpha v + \sum_{i=1}^{m} y_i^2 + \sum_{i=1}^{m} \frac{\partial v}{\partial y_i} Y_i (y_1, \ldots, y_m, z_1, \ldots, z_p)$$

In accordance with the condition (1.4) of Lemma 3 in [4] the derivative dv / dt is a function, positive-definite with respect to the variables $y_1, \ldots, y_m$ in the region $v > 0$, i.e. according to the theorem given in [5], the unperturbed motion is unstable with respect to the variables $y_1, \ldots, y_m$.

Theorem 3. If the equation (1.5) has no roots with positive real parts but has roots with real parts equal to zero, then with Condition A holding, the higher order terms in the first group of equations (1.3) can be chosen so as to obtain satbility or instability with respect to the variables $y_1, \ldots, y_m$.

Thus we see that the cases which may arise in the investigation of the problem of stability with respect to the variables $y_1, \ldots, y_m$ when the equations of perturbed motion have the form (1.3), can be divided into two categories, namely, the noncritical cases when the problem is solved using the first approximation equations (Theorems 1 and 2) and the critical cases when higher order terms must be taken into account (Theorem 3).

We shall consider the simplest critical case, that of a single zero root.

2. Let us investigate the stability of the unperturbed motion with respect to the variables $y_1, \ldots, y_m$ of the system (1.3) under the assumption that one root of (1.5) is equal to zero and the remaining roots have negative real parts, i.e. we shall consider the critical case of a single zero root while solving the problem of stability with respect to a part of the variables.

We transform the system of equations

$$\frac{dy_i}{dt} = \sum_{j=1}^{m} a_{ij} y_j, \qquad i = 1, \ldots, m$$

to the form [4]

$$\frac{du}{dt} = 0, \quad \frac{du_i}{dt} = \sum_{j=1}^{m-1} p_{ij} u_j + p_i u, \quad i = 1, \ldots, m-1$$

Then the system (1.3) becomes

$$\frac{du}{dt} = U(u, u_1, \ldots, u_{m-1}, z_1, \ldots, z_p) \tag{2.1}$$

$$\frac{du_i}{dt} = \sum_{j=1}^{m-1} p_{ij} u_j + p_i u + U_i(u, u_1, \ldots, u_{m-1}, z_1, \ldots, z_p)$$

$$\frac{dz_k}{dt} = \sum_{j=1}^{m-1} b_{kj}^* u_j + b_k^* u + \sum_{j=1}^{p} c_{kj} z_j + Z_k^*(u, u_1, \ldots, u_{m-1}, z_1, \ldots, z_p), \quad i = 1, \ldots, m-1; \quad k = 1, \ldots, p$$

where conditions of the type (1.4) hold for the functions U and U_i ($i = 1, \ldots, m-1$). Next we consider the system of equations

$$f_i = \sum_{j=1}^{m-1} p_{ij} u_j + p_i u + U_i(u, u_1, \ldots, u_{m-1}, z_1, \ldots, z_p) = 0 \tag{2.2}$$

$$i = 1, \ldots, m-1$$

defining the variables u_i as functions of the variables u and $z_1, \ldots, z_p$. When $u = u_1 = \ldots = u_{m-1} = 0$, the functional determinant in the variables $u_1, \ldots, u_{m-1}$ of this system is not zero [4], therefore a unique solution of the system (2.2) exists of the type

$$w_j = u_j(u, z_1, \ldots, z_p) = A_j^{(1)}(z_1, \ldots, z_p)\, u + A_j^{(2)}(z_1, \ldots, z_p)\, u^2 + \ldots, \qquad j = 1, \ldots, m-1$$

and the coefficients $A_j^{(1)}(z_1, \ldots, z_p)$ are constant.

Let us now transform the variables in (2.1) as follows:

$$u_j = \xi_j + w_j, \qquad j = 1, \ldots, m-1$$

We obtain

$$du / dt = \bar{U}(u, \xi_1 \ldots, \xi_{m-1}, z_1, \ldots, z_p) \tag{2.3}$$

$$\frac{d\xi_i}{dt} = \sum_{j=1}^{m-1} p_{ij} \xi_j + \bar{U}_i(u, \xi_1, \ldots, \xi_{m-1}, z_1 \ldots, z_p)$$

$$\frac{dz_k}{dt} = \sum_{j=1}^{m-1} b_{kj}^* \xi_j + b_k^* u + \sum_{j=1}^{p} c_{kj} z_j + \sum_{j=1}^{m-1} b_{kj}^* w_j + Z_k^*(u, \xi_1 + w_1, \ldots, \xi_{m-1} + w_{m-1}, z_1, \ldots, z_p),$$

$$i = 1, \ldots, m-1; \quad k = 1, \ldots, p$$

where

$$\bar{U}(u, \xi_1, \ldots, \xi_{m-1}, z_1, \ldots, z_p) = U(u, \xi_1 + w_1, \ldots, \xi_{m-1} + w_{m-1}, z_1, \ldots, z_p)$$

$$\bar{U}_i(u_1, \xi_1, \ldots, \xi_{m-1}, z_1, \ldots, z_p) =$$

$$U_i(u, \xi_1 + w_1, \ldots, \xi_{m-1} + w_{m-1}, z_1, \ldots, z_p) -$$
$$U_i(u, w_1, \ldots, w_{m-1}, z_1, \ldots, z_p) - \frac{\partial w_i}{\partial u}\frac{du}{dt} - \sum_{j=1}^{p} \frac{\partial w_i}{\partial z_j}\frac{dz_j}{dt}$$

are analytic functions of the variables u, ξ_i and z_k; the expansion of these functions begins with the terms of at least second order. Since the new variables vanish simultaneously if and only if the old variables do so, the problem of stability with respect to one set of the variables is equivalent to that in the other variables. For this reason we can utilize (2.3) in the process of solving the problem. Let us denote

$$\bar U^{(0)}(u, z_1, \ldots, z_p) = \bar U(u, 0, \ldots, 0, z_1, \ldots, z_p) =$$
$$g(z_1, \ldots, z_p) u^\alpha + g_1(z_1, \ldots, z_p) u^{\alpha+1} + \ldots$$
$$\bar U_i^{(0)}(u, z_1, \ldots, z_p) = \bar U_i(u, 0, \ldots, 0, z_1, \ldots, z_p) =$$
$$h(z_1, \ldots, z_p) u^\beta + h_1(z_1, \ldots, z_p) u^{\beta+1} \ldots, \quad i = 1, \ldots, m-1$$

and let

$$\bar U^{(0)} \not\equiv 0, \quad \beta > \alpha, \qquad g(z_1, \ldots, z_p) = g_0 + g^*(z_1, \ldots, z_p) \tag{2.4}$$

where g_0 is a constant. Let also the functions

$$\bar U(0, \xi_1, \ldots, \xi_{m-1}, z_1, \ldots, z_p), \ \bar U_i(0, \xi_1, \ldots, \xi_{m-1}, z_1, \ldots, z_p)$$
$$(i = 1, \ldots, m-1)$$

contain no terms linear in $\xi_1, \ldots, \xi_{m-1}$ (Condition B). We take the Liapunov function in the form

$$v = \frac{1}{\alpha+1} g_0 u^{\alpha+1} - v_2$$

where $v_2 = v_2(\xi_1, \ldots, \xi_{m-1})$ is a positive-definite quadratic form such that

$$\sum_{i=1}^{m-1} \frac{\partial v_2}{\partial \xi_i}\Big(\sum_{j=1}^{m-1} p_{ij}\xi_j\Big) = -\sum_{i=1}^{m-1} \xi_i^2$$

We find the derivative of v using the equations (2.3) of perturbed motion

$$\frac{dv}{dt} = g_0 g u^{2\alpha} + g_0 g_1 u^{2\alpha+1} + \ldots + \tag{2.5}$$
$$g_0 u^\alpha \sum_{i=1}^{m-1} \xi_i F_i(u, \xi_1, \ldots, \xi_{m-1}, z_1, \ldots, z_p) + \sum_{i=1}^{m-1} \xi_i^2 +$$
$$\sum_{i,j=1}^{m-1} \xi_i\xi_j F_{ij}(u, \xi_1, \ldots, \xi_{m-1}, z_1 \ldots, z_p) + u^\alpha \sum_{i=1}^{m-1} \xi_i F_i^*(u, z_1, \ldots, z_p)$$

where by virtue of the Conditions B and (1.4), we have

$$F_i(0, 0, \ldots, 0, z_1, \ldots, z_p) = 0, \ F_{ij}(0, 0, \ldots, 0, z_1, \ldots, z_p) = 0$$
$$F_i^*(0, z_1, \ldots, z_p) = 0$$

and where the expansion of all functions begins with first order terms.

Remembering that Condition A holds for $z_j (j = 1, \ldots, p)$, we now consider two cases. In the first case the function $g(z_1, \ldots, z_p)$ assumes only positive values for any z_j, i.e. g_0 is a positive constant and $g^*(z_1, \ldots, z_p)$ is a positive-definite function. Then dv/dt is a positive-definite function of the variables $u, \xi_1, \ldots, \xi_{m-1}$, consequently, the unperturbed motion of the system (2.3) is unstable with respect to the variables $u, \xi_1, \ldots, \xi_{m-1}$ [5]. In the second case the function $g(z_1, \ldots, z_p)$ assumes only negative values for any z_j. Then the derivative (2.5) is a positive-definite function in the variables $u, \xi_1, \ldots, \xi_{m-1}$ and it is evident that the unperturbed motion is asymptotically stable with respect to the variables $u, \xi_1, \ldots, \xi_{m-1}$ when α is odd, and unstable when α is even [5].

Thus, we obtain the following theorem.

Theorem 4. Let the equations of perturbed motion used in investigating the stability with respect to a part of the variables be reduced in the critical case of a single zero root to the form (2.3). Let also the Conditions A, B (1.4) and (2.4) all hold. Then, if $g(z_1, \ldots, z_p)$ assumes only the negative values, the unperturbed motion of the system (2.3) is asymptotically y-stable if α is odd, and y-unstable if α is even; if $g(z_1, \ldots, z_p)$ assumes only the positive values, then the unperturbed motion of the system (2.3) is y-unstable.

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Translated by L. K.

UDC 538.4

ON THE STABILITY OF A LAMINAR FLOW OF A CONDUCTING FLUID FILM IN A TRANSVERSE ELECTRIC FIELD

PMM Vol. 39, №3, 1975, pp. 427-433

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(Received April 3, 1974)

Averaged equations for defining in long-wave approximation the flow of a conducting fluid film on the vertical wall of a plane channel in a transverse electric field are derived. The convective and absolute instability of laminar flow is investigated with the use of these equations. It is established that periodic perturbations are intensified downstream of the flow only if their frequency does not exceed the critical frequency which depends on the electrohydrodynamic interaction parameter and the Weber number. It is shown that the electric field has a destabilizing effect owing to the increase of surface charge density in the vicinity of wave crests. This results in an increase of surface forces produced by the electric field at wave crests thereby reducing the stabilizing effect of surface tension forces. Proof is given of the absolute stability of the laminar flow of a